A method of directly defining the inverse mapping for solutions of cauchy reaction-diffusion problems

B.S.K. De Silva[1*]

Postgraduate Institute of Science University of Peradeniya Peradeniya, Sri Lanka
sahankalana95@gmail.com

M.T.M. Dewasurendra[2]

Department of Mathematics Faculty of Science University of Peradeniya Peradeniya, Sri Lanka

mangalagama.dewasurendra@sci.pdn.ac.lk

***Abstract*—In this paper, a semi-analytical approach called the Method of Directly Defining inverse Mapping (MDDiM) is applied to obtain an approximation series solution for the time-dependent reaction-diffusion problems. This novel approach is being applied to reaction-diffusion problems for the first time. Reaction-diffusion equations are used to describe, in modeling, nonlinear systems across various fields, including physics, chemistry, ecology, biology, and engineering. Traditionally, perturbation and asymptotic methods were widely used to obtain analytical approximations for nonlinear problems. However, these methods often failed under strong nonlinearity conditions. To overcome this obstacle, Shijun Liao introduced the Homotopy Analysis Method (HAM) in 1992, which is effective for highly nonlinear problems. However, determining the unknown functions requires computing the inverse of the linear operator. In scientific computing, calculating the inverse operator for a differential equation is a time-consuming process. To overcome this obstacle, in 2016 Liao introduced a new approach, namely the Method of Directly Defining inverse Mapping (MDDiM) with the freedom to choose the inverse linear mapping directly. The proposed method achieved a remarkable level of accuracy for solutions even with few iterations. Therefore, it can be concluded that MDDiM is not only easy to use but also accurate.**

Keywords— ***Cauchy reaction-diffusion problems, Method of directly defining inverse mapping, Homotopy analysis method***

VI. INTRODUCTION

One-dimensional time-dependent reaction-diffusion equation is expressed as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + R(u) + g(x,t), \quad a < x < b, \quad t > 0 \tag{1}$$

with the initial condition

$$u(x,0) = f(x), \quad a \le x \le b \tag{2}$$

and the boundary conditions

$$u(a,t) = h_1(t), \quad u(b,t) = h_2(t), \quad t > 0 \tag{3}$$

where $R(u)$ is a nonlinear function of u, $g(x,t)$, $f(x)$, $h_1(t)$, and $h_2(t)$ are known functions, and $u(x,t)$ is the unknown function to be determined. Reaction-diffusion equations are used to describe, in modeling, nonlinear systems in fields such as physics, chemistry, ecology, biology, and engineering [1-4]. Lesnic [5] provided approximate series solutions for reaction- diffusion problems using the analytic Adomian Decomposition Method (ADM). Dehghan and Shakeri

[6] also employed the Variational Iteration Method (VIM) to obtain similar approximate solutions for these problems. Bataineh, Noorani, and Hashim [7] used the Homotopy Analysis Method (HAM) to solve Cauchy reaction-diffusion problems. In 2016, the Method of Directly Defining inverse Mapping (MDDiM) was introduced by Liao [8] to solve a nonlinear ordinary differential equation. The Method of Directly Defining inverse Mapping builds upon the Homotopy Analysis Method [9-12], an analytical approximation technique used for solving highly nonlinear differential equations. This method allows the freedom to choose the inverse linear mapping directly, avoiding the need to calculate the inverse of an operator. Recently, the MDDiM was extended by Vajravelu and his research group [13-17] to systems of nonlinear ordinary differential equations. Also, this novel technique was further developed to solve single and coupled nonlinear PDEs [18-19]. In this study, we further applied this novel method to address Cauchy reaction-diffusion problems. All computational results were achieved using Maple 16.

VII. MATERIAL AND METHODS

First of all, we describe the basic ideas of the MDDiM.

Consider the following nonlinear partial differential

$$\frac{\partial}{\partial t}(x,t) = D\frac{\partial}{\partial x^2}(x,t) + r(x,t)u(x,t),$$

$$(x,t) \in \Omega \subset \Re^2,$$

(1)

equation,

$$\mathcal{N}[u(\bar{x})] = 0\,, \quad \bar{x} \in \Omega\,,$$

(4)where u is the concentration, r is the reaction parameter and $D > 0$ is the diffusion coefficient. The initial and boundary conditions are

subject to μ linear boundary conditions independent. All functions that are expressed by S_∞

form a set of functions, denoted by $V = \sum_{k=0}^{+\infty} a_k \varphi_k(\bar{x})$. The approximate solution comes from

this space. Next define the space for the initial guess

$V^* = \sum^{\mu}$ of $a\ \varphi\ (\bar{x})$, taking the linear combination

$k=0$ k k

first μ functions of the set S_∞ , and $\hat{V}$ =

$\sum^{+\infty}$ $b\ \varphi\ (\bar{x})$ so that $V = V^* + \hat{V}$. Assume that

$k=\mu+1$ k k

$u(\bar{x}) \in V$ and the μ unknown coefficients can be

uniquely determined by the μ linear boundary conditions (5). Similarly, letting $S_R =$

$\{\psi_1(\bar{x}), \psi_2(\bar{x}), \dots\}$ and then define $U = \sum_{k=0}^{+\infty} c_k \psi_k(\bar{x})$ so that $\mathcal{N}[u(\bar{x})] \in U$. Finally directly defining the inverse mapping $\Phi: U \to V$. We obtain the

following higher order deformation equations for the

MDDiM [8,19]:

$$\mathfrak{B}_i[u(\bar{x})] = \beta_i\,, \text{ at } \bar{x} = \bar{\alpha_i}\,, \qquad i = 1,2,3,\dots,\mu.$$

(5)

$$u(x,0) = f(x), \quad x \in \Re,$$

$$u(0,t) = g_0(t), \qquad \frac{\partial u}{\partial x}(0,t) = g_1(t), \quad t \in \Re.$$

Example 1. Case r = constant

Taking $D = 1$ and $r = -1$, problem (1)-(3) becomes:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - u(x,t), \quad (x,t) \in \Omega \subset \Re^2, \tag{9}$$

subject to the initial and boundary conditions

$$u(x,0) = e^{-x} + x = f(x), \quad x \in \Re, \tag{10}$$

$$u(0,t) = 1 = g_0(t), \tag{11}$$

$$\frac{\partial}{\partial x}(0,t) = e^{-t} - 1 = g_1(t), \quad t \in \Re. \tag{12}$$

Then, it was straight forward to choose initial guess

$u_0(x,t) = e^{-x} + x$. In the frame work of the MDDiM, we directly defined the inverse mapping, $\Phi: U \to V$

$$\Phi[t^{k+1}] = \frac{t^k}{Ak+1} \tag{13}$$

$$u_k(\bar{x}) = \chi_k u_{k-1}(\bar{x}) + c_0 \Phi[\delta_{k-1}(\bar{x})] + \sum_{n=1}^{\mu} a_{k,n}\varphi_n(\bar{x}). \quad (6)$$

where A is constant to be selected. The special solutions is; $k=0$

For the error analysis, we first consider the n-term solution (or the approximate series solution) $u(\bar{x}) =$

$$u_k(x, t) = \chi_k u_{k-1}(x, t) + c_0 \Phi[\mathcal{D}_{k-1}] + a_{k,0},$$

where $a_{k,0}$ determined by $u_k(x, 0) = 0$.

$u_0(\bar{x}) + \sum_{k=1}^{n-1} u_k(\bar{x})$, using (6) and choosing an appropriate initial guesses $u_0(\bar{x})$. Then, by adding the first n terms of the series solution, we define square. uniquely determined by the μ linear boundary conditions (5).

Similarly, letting $S_R =$ $\{y_1(\bar{x}), y_2(\bar{x}), \ldots\}$ and then define U

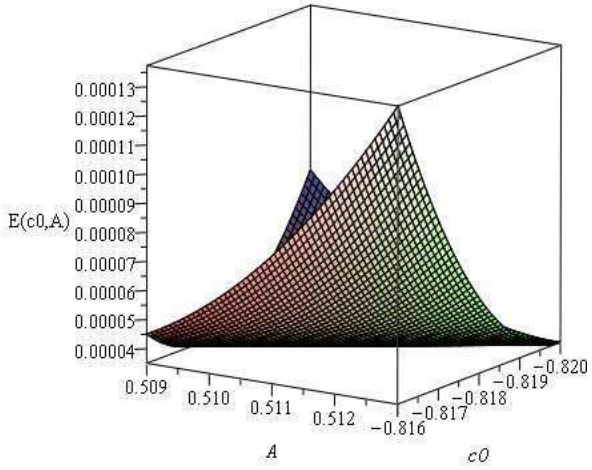
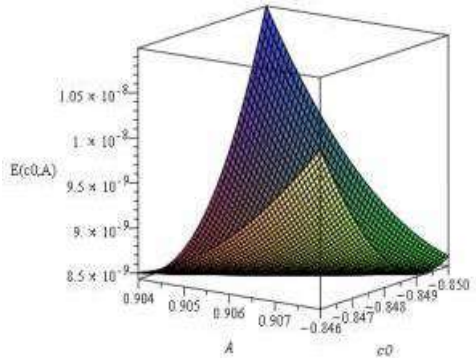


Fig. 2 Graph of the approximations solution and exact



solution for example 1.

A. Example 2. Case $r = r(t)$

Taking $D = 1$ and $r(t) = 2t$, problem (1)-(3) becomes:

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + 2tu(x, t), \quad (x, t) \in \Omega \subset \Re^2,$$

subject to the initial and boundary conditions

$$u(x, 0) = e^x = f(x), \quad x \in \Re,$$

$$u(0, t) = e^{t+t^2} = g_0(t),$$

$$\frac{\partial u}{\partial x}(0, t) = e^{t+t^2} = g_1(t), \quad t \in \Re.$$

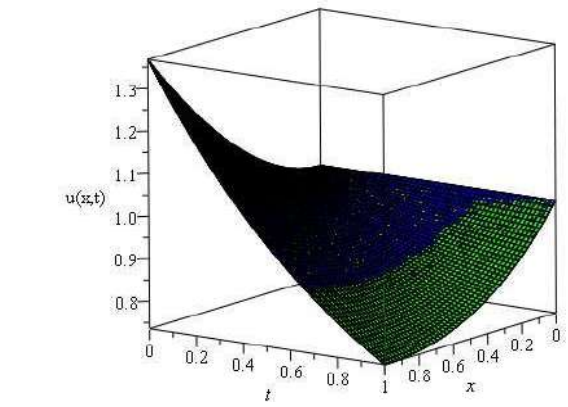


Fig. 3. Plot of $E(c_0, A)$, the squared residual error as a function of c_0 and A for example 2. The error function has minimum

$E(c_0, A) = 3.5003 \times 10^{-5}$ where $c_0 = -0.8189$ and $A = 0.5113$.

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Then, it was straight forward to choose initial guess

$u_0(x, t) = e^x$. In the frame work of the MDDiM, the same inverse mapping (13) was used.

The special solutions is

$$u_k(x, t) = \chi_k u_{k-1}(x, t) + c_0 \Phi[\mathcal{D}_{k-1}] + a_{k,0}, \quad (20)$$

where $a_{k,0}$ determined by $u_k(x, 0) = 0$.

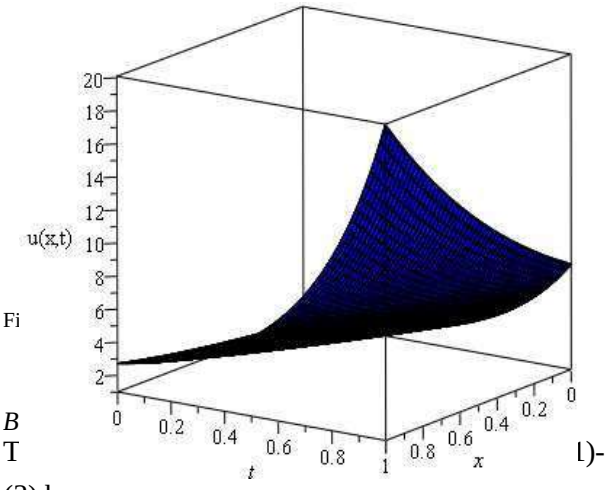
Using only five terms, let $u(x, t) = u_0(x, t) +$

$u_1(x, t) + u_2(x, t) + u_3(x, t) + u_4(x, t)$. The sum of

squared residual error functions is given by

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(3) becomes:

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) - (1 + 4x^2)u(x, t), \quad (22)$$

$(x, t) \in \Omega \subset \Re^2$,

subject to the initial and boundary conditions

$$E(c_0, A) = \frac{}{250000} \sum \sum (\mathcal{N}[\varphi($$

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$$f(x), \quad x \in \Re,$$

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CONCLUSIONS

In this paper, MDDiM was developed and applied to solve Cauchy reaction-diffusion problems, which are common in various scientific phenomena. Since the direct definition of the inverse operator, the series solutions were obtained using less CPU time, low error, and less complicated terms. The proposed technique produces a highly accurate and reliable solution to the problems in a few iterations. Therefore, it can be concluded that MDDiM is not only easy to use but also accurate.

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